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MONOTONICITY OF PERTURBED TRIDIAGONAL M-MATRICES*

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Abstract. A well-known property of an M-matrix is that its inverse is elementwise nonnegative, which we write as $M^{-1} \geq 0$. In a previous paper [*Linear Algebra Appl.*, 434 (2011), pp. 131–143], we gave sufficient bounds on single element perturbations so that monotonicity persists for a perturbed tridiagonal M-matrix. Here we extend these results, presenting the *actual* maximum upper bounds on single element perturbations, as well as sufficient and necessary conditions for the maximum allowable higher rank perturbations. Perturbed Toeplitz tridiagonal M-matrices are considered as a special case. We compare our results to existing normwise bounds due to Bouchon and an iterative algorithm provided by Buffoni. We demonstrate the utility of these results by considering an application: ensuring a nonnegative solution of a discrete analogue of an integro-differential population model.

Key words. M-matrix, tridiagonal, inverse positivity, monotone, perturbation

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1. Introduction.

M-matrices arise in many areas of mathematics and statistics, including the convergence analysis of iterative processes for the solution of linear systems, the analysis of schemes for the numerical solution of differential equations (the discrete maximum principle and convergence analysis, for instance [27]), the study of stability of economic equilibria [12], and the analysis of Markov chains [19]. This class of matrices was apparently introduced and named by Ostrowski [23], most likely in reference to the work of Minkowski [20, 21]. There are a great many different, but not obviously equivalent, characterizations of M-matrices. Berman and Plemmons [2] give at least 50 such conditions as well as an extensive bibliography.

A Z-matrix is a square matrix whose off-diagonal entries are nonpositive [11]. For our purposes, we will say a matrix M is a nonsingular M-matrix if and only if M is a nonsingular Z-matrix with positive diagonal entries and the inverse of M is nonnegative; that is, M is monotone. M-matrices are a subset of a larger class of monotone matrices which enjoy nonnegative inverses. Here the nonnegativity of the inverse is assumed to hold in an elementwise fashion. An equivalent condition is that M is an M-matrix if and only if M is a Z-matrix with the positive diagonal entries and MD is strictly row diagonally dominant for some positive diagonal matrix D; i.e., M is generalized strictly diagonally dominant.

Here we are interested in particular elementwise perturbations of tridiagonal M-matrices; specifically we attempt to understand under what conditions the nonnega-

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tive inverse persists. In other words, we would like to determine whether the perturbed M-matrix is monotone. Primarily we consider the effect of perturbations outside the tridiagonal band. Clearly if the perturbed matrix is a Z-matrix (that is, the element perturbations are negative), which maintains the necessary diagonal dominance, then the nonnegative inverse will persist. Hence we will restrict ourselves to positive elementwise perturbations outside the tridiagonal band. Such matrices are no longer M-matrices but may indeed be monotone. The persistence of a nonnegative inverse has been studied in the past [6, 5, 4, 28, 3, 13, 16, 15], both as an interesting linear algebra problem and motivated by particular applications.

Bramble and Hubbard [5, 4] consider higher order finite difference solutions of a second order linear elliptic boundary value problem subject to Dirichlet boundary conditions. The solutions involve a linear system matrix, which, in spite of not being an M-matrix (or a Z-matrix for that matter), is indeed shown to be monotone. Zafarullah [28] extends the work of Bramble and Hubbard to include problems with Neumann or third kind boundary conditions. Again the M-matrix structure is lost but monotonicity is preserved. Buffoni [6] presents an algorithm which is numerically able to find the maximum allowable positive perturbation of an inverse nonnegative matrix so that the monotonicity property persists. The result is applied to the study of a discrete analogue of an integro-differential equation population model. The algorithm is used to determine the maximum value of a model parameter so that the solution is positive (hence physically relevant) and stable. We will revisit Buffoni's work and his application problem in sections 4 and 5. Bouchon [3] considers matrix perturbations of M-matrices and provides a computable upper bound on the norm of the perturbation to ensure the perturbed matrix is monotone. The result is used to prove convergence of a noncentered finite difference approximation to a second order boundary value problem. We will compare our result to Bouchon's work in sections 3.2, 4, and 5.

More recently, the persistence of monotonicity subject to elementwise perturbations, which destroy the M-matrix property, has been studied in a sequence of three papers [13, 16, 15]. Using the Sherman–Morrison formula [25] to explicitly express the inverse of the perturbed matrix, and decay estimates, which characterize the entries of symmetric tridiagonal M-matrices (see section 2), Haynes, Trummer, and Kennedy [13] obtained sufficient upper bounds for positive perturbations just outside the tridiagonal band. A sufficient upper bound was also derived for a particular single element perturbation of nonsymmetric, tridiagonal M-matrices. These results were extended by Kennedy and Haynes in [16], where sufficient upper bounds were derived for an arbitrary single element perturbation of a nonsymmetric, tridiagonal M-matrix.

In [15], we use explicit representations, in terms of determinants, for elements of inverse matrices to obtain improved sufficient bounds for the maximum allowable single element perturbations outside the tridiagonal band. The bounds are again written in terms of decay rates characterizing the inverse of the original M-matrix and as such are qualitatively similar to those found in [13, 16]. In the case of perturbations to elements on the second super- and subdiagonals, these upper bounds are also shown to be the actual maximum allowable perturbations.

In this paper, we extend the results of [13, 16, 15] and provide necessary *and* sufficient bounds on elementwise perturbations of nonsymmetric, tridiagonal M-matrices. Moreover, these bounds are no longer provided in terms of the decay estimates which characterize the decay rate of M^{-1}, but in terms of the first sub- and superdiagonals of M (elements $(l, l \pm 1)$) and the principal minors of the unperturbed matrix. We provide results for Toeplitz tridiagonal M-matrices as a special case. We also consider the effect of many simultaneous elementwise perturbations and obtain necessary and sufficient bounds for the persistence of monotonicity.

The outline of the paper is as follows. Section 2 reviews several known properties of tridiagonal M-matrices and their inverses. In section 3 we obtain the actual maximum allowable perturbation for the inverse positivity of an M-matrix subject to an arbitrary single element perturbation. For comparison, we present the results of Bouchon [3], and through a series of simple examples we demonstrate our improved result. Section 4 details the sufficient and necessary conditions for higher rank perturbations generated by many simultaneous elementwise perturbations. We demonstrate the efficacy of our results by considering the application problem of Buffoni in section 5.

2. Properties of tridiagonal *M*-matrices. We begin by establishing some useful notation. For any matrix $X_n = (x_{ij}) \in \mathbb{R}^{n\times n}$, let $X_k, k = 1,\dots,n$, be the leading principal submatrix of X_n of order k. Also, let $X_{[l:k]}$, $1 < l \le k \le n$, denote the principal submatrix consisting of rows and columns l through k of X_n, i.e.,

$$X_{[l:k]} = \begin{pmatrix} x_{ll} & \dots & x_{lk} \\ \vdots & & \vdots \\ x_{kl} & \dots & x_{kk} \end{pmatrix}.$$

Moreover, if X_n is invertible, $(X_n^{-1})_{ij}$ will denote the (i,j) element of X_n^{-1}, and we write $X_n^{-1} \ge 0$ if X_n^{-1} is elementwise nonnegative. In addition, let $E_{i,j}$ be the n-by-n matrix whose (i,j) entry is 1 while all other entries are 0. As is usual, for any vector $\alpha \in \mathbb{R}^n$, α^T will denote the transpose of α.

In this paper we consider elementwise perturbations of a general tridiagonal M-matrix

$$M_n = \begin{pmatrix} a_1 & -c_1 & 0 & \dots & 0 \\ -b_1 & a_2 & -c_2 & & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & & -b_{n-2} & a_{n-1} & -c_{n-1} \\ 0 & \dots & 0 & -b_{n-1} & a_n \end{pmatrix}, \tag{2.1}$$

where a_i, b_i, and $c_i > 0$. Any required diagonal dominance assumptions will be noted as needed.

M-matrices have positive principal minors. We detail this result, used throughout the paper, in Lemma 2.1 (see [14]).

LEMMA 2.1. *A Z-matrix M_n is an M-matrix if and only if* $\det M_k > 0$ *for* $k = 1,\dots,n$. *Furthermore, a Z-matrix M_n is an M-matrix if and only if* $\det M_{[l:k]} > 0$ *for* $1 \le l \le k \le n$.

M-matrices need not be diagonally dominant; however, the following lemma (again, see [14]) shows that an M-matrix is closely related to a strictly diagonally dominant matrix.

LEMMA 2.2. *A Z-matrix M_n is an M-matrix if and only if M_n has positive diagonal entries and there exists positive diagonal matrices X_n and Y_n such that $Y_nM_nX_n$ is strictly row and column diagonally dominant.*

The inverses of banded matrices, including estimates of the entries of M_n^{-1}, have been studied and characterized by many authors [1, 8, 10, 9]. The special case of tridiagonal M-matrices has received additional attention. In particular, short term recurrences are available which allow explicit calculation of elements of the inverse [7, 17, 18]. In addition to the monotonicity of M_n, the inverses of row diagonally dominant tridiagonal M-matrices are known to have special structure: the largest elements in each column occur along the diagonal, and the elements in each column decay from the diagonal in a way which can be quantified as in Lemma 2.3. This result for the general nonsymmetric tridiagonal case, originally due to Nabben [22], was later refined for M-matrices by Peluso and Politi [24].

LEMMA 2.3. *Let M_n be a row diagonally dominant tridiagonal M-matrix. If $(M_n^{-1})_{ij}$ is the (i,j) element of M_n^{-1}, then*

$$\delta_i(M_n^{-1})_{i+1,j} \leq (M_n^{-1})_{ij} \leq \tau_i(M_n^{-1})_{i+1,j}, \quad i = 1, \dots, j-1,$$

and

$$\gamma_i(M_n^{-1})_{i-1,j} \leq (M_n^{-1})_{ij} \leq \omega_i(M_n^{-1})_{i-1,j}, \quad i = j+1, \dots, n,$$

where the decay parameters are given by

$$\delta_i = \frac{c_i}{a_i}, \ \tau_i = \frac{c_i}{a_i - b_{i-1}}, \ \gamma_i = \frac{b_{i-1}}{a_i}, \text{ and } \omega_i = \frac{b_{i-1}}{a_i - c_i},$$

and we define $b_0 = c_n = 0$ for consistency.

The decay factors depend only on the (row) position in a column, not on the particular column itself. If M_n is symmetric, then we have the same decay along columns and rows; in this case $\delta_i = \gamma_i$ and $\tau_i = \omega_i$. We note that Nabben [22] provides an iterative algorithm to improve the decay parameters above. The maximum allowable elementwise perturbation bounds found in [13, 16, 15] are expressed in terms of these decay parameters.

An immediate, useful consequence of Lemma 2.3, obtained by choosing $j = n$ and $j = 1$, is given in Corollary 2.4.

COROLLARY 2.4. *Let M_n be a row diagonally dominant tridiagonal M-matrix. If $(M_n^{-1})_{ij}$ is the (i,j) element of M_n^{-1}, then*

$$(M_n^{-1})_{in} \leq (M_n^{-1})_{i+1,n}, \quad (M_n^{-1})_{i1} \geq (M_n^{-1})_{i+1,1}$$

for $i = 1, \dots, n-1$.

3. Single element perturbations. In this section, we explore the persistence of monotonicity for a tridiagonal M-matrix subject to a single element perturbation. It has been pointed out in [16] that if a perturbation does not change the M-matrix sign pattern, then a sufficient condition to ensure the nonnegativity of the inverse is obtained by imposing the required diagonal dominance property. Moreover, when perturbing an element inside the diagonal band, it is not possible to maintain the nonnegative inverse property without also maintaining the M-matrix sign pattern. Hence, we restrict ourselves to single element perturbations outside the tridiagonal band which destroy the M-matrix sign pattern. Sufficient upper bounds on such an arbitrary single element perturbation were given in [16, 15]. Here, we extend these results and present the actual, necessary, and sufficient maximum allowable perturbations. Frequently occurring Toeplitz matrices are considered as a special case. We then compare the results to these previous findings and the work of Bouchon [3].

3.1. Theoretical results. We begin by presenting a useful lemma which extends Theorem 1 from [15].

LEMMA 3.1. *Let M_n be a tridiagonal M-matrix. Let $P_n = M_n + hE_{l,k}$, where $h \geq 0$, and $|l-k| \geq 2$. Let P_{k_0} be the k_0-by-k_0 leading principal submatrix of P_n, where $k_0 = \max\{k, l\}$. Then P_n^{-1} is elementwise nonnegative if and only if $P_{k_0}^{-1}$ is elementwise nonnegative.*

Proof. Without loss of generality, we consider only $l \leq k-2$ (and hence $k_0 = \max\{k,l\} = k$), as the case $l \geq k+2$ follows by a similar analysis.

We first show that if $P_{k_0}^{-1} = P_k^{-1} \geq 0$, then $P_n^{-1} \geq 0$. The result holds naturally for $k = n$. For $k = 3, \ldots, n-1$, we first argue that if $P_k^{-1} \geq 0$, then $P_{k+1}^{-1} \geq 0$. Partition P_{k+1} as

$$P_{k+1} = \begin{pmatrix} P_k & -\alpha_k \\ -\beta_k^T & a_{k+1} \end{pmatrix},$$

where $\alpha_k = (0, \ldots, 0, c_k)^T$ and $\beta_k = (0, \ldots, 0, b_k)^T$ are elementwise nonnegative vectors of length k. Assume $P_k^{-1} \geq 0$; then both $P_k^{-1}\alpha_k$ and $\beta_k^T P_k^{-1}$ are nonnegative. Moreover, define $\rho_{k+1} = \frac{\det P_{k+1}}{\det P_k}$ and $\det M_0 = 1$; then a cofactor expansion gives

$$\rho_{k+1} = \frac{\det P_{k+1}}{\det P_k} = \frac{\det M_{k+1} + h \det M_{l-1} \prod_{t=l}^{k-1} b_t a_{k+1}}{\det M_k + h \det M_{l-1} \prod_{t=l}^{k-1} b_t} > 0.$$

Thus, we obtain

$$P_{k+1}^{-1} = \begin{pmatrix} P_k^{-1} + \frac{1}{\rho_{k+1}}(P_k^{-1}\alpha_k)(\beta_k^T P_k^{-1}) & \frac{1}{\rho_{k+1}} P_k^{-1}\alpha_k \\ \frac{1}{\rho_{k+1}}\beta_k^T P_k^{-1} & \frac{1}{\rho_{k+1}} \end{pmatrix} \geq 0.$$

Similarly, $P_{k+1}^{-1} \geq 0$ leads to $P_{k+2}^{-1} \geq 0$. Inductively, we have $P_n^{-1} \geq 0$.

We now show that $P_n^{-1} \geq 0$ implies $P_{k_0}^{-1} = P_k^{-1} \geq 0$. A direct calculation shows

$$\det P_s = \det M_s + h \det M_{l-1} \prod_{t=l}^{k-1} b_t \det M_{[k+1:s]} > 0$$

for $s = k, \ldots, n$, where we have defined $\det M_{[k+1:k]} = \det M_0 = 1$. Moreover, for $s = k+1, \ldots, n$, define $\rho_s = \frac{\det P_s}{\det P_{s-1}} > 0$; then we have

$$P_s^{-1} = \begin{pmatrix} P_{s-1}^{-1} + \frac{1}{\rho_s}(P_{s-1}^{-1}\alpha_{s-1})(\beta_{s-1}^T P_{s-1}^{-1}) & \frac{1}{\rho_s} P_{s-1}^{-1}\alpha_{s-1} \\ \frac{1}{\rho_s}\beta_{s-1}^T P_{s-1}^{-1} & \frac{1}{\rho_s} \end{pmatrix}, \tag{3.1}$$

where $\alpha_{s-1} = (0, \ldots, 0, c_{s-1})^T$ and $\beta_{s-1} = (0, \ldots, 0, b_{s-1})^T$. If $P_n^{-1} \geq 0$, then using (3.1) with $s = n$ implies $P_{n-1}^{-1}\alpha_{n-1} \geq 0$. This says that the last column of P_{n-1}^{-1} is nonnegative. Thus, $P_{n-2}^{-1}\alpha_{n-2} \geq 0$ from (3.1) with $s = n-1$. Recursively, we have $P_k^{-1}\alpha_k \geq 0$. It follows that the last column of P_k^{-1} is nonnegative.

In fact, partition P_k as

$$P_k = \begin{pmatrix} M_{k-1} & -\alpha_{k-1} \\ -\beta_{k-1}^T & a_k \end{pmatrix}; \tag{3.2}$$

then

$$P_k^{-1} = \begin{pmatrix} M_{k-1}^{-1} + \frac{1}{\rho_k}(M_{k-1}^{-1}\alpha_{k-1})(\beta_{k-1}^T M_{k-1}^{-1}) & \frac{1}{\rho_k} M_{k-1}^{-1}\alpha_{k-1} \\ \frac{1}{\rho_k}\beta_{k-1}^T M_{k-1}^{-1} & \frac{1}{\rho_k} \end{pmatrix},$$

where

$$\alpha_{k-1} = (0, \dots, 0, -h, 0, \dots, 0, c_{k-1})^T, \quad \beta_{k-1} = (0, \dots, 0, b_{k-1})^T,$$

and $\rho_k = \det P_k / \det M_{k-1} > 0$. Clearly, the nonnegativity of the last column ensures the nonnegativity of P_k^{-1}. ∎

We now obtain necessary and sufficient upper bounds on a single element perturbation in position (l, k) so that $P_{k_0}^{-1} \geq 0$. Lemma 3.1 suggests that this is the essential step for ensuring the monotonicity of P_n.

LEMMA 3.2. *Let M_n and P_{k_0} be defined as in Lemma 3.1. $P_{k_0}^{-1}$ is elementwise nonnegative if and only if h satisfies*

$$h \leq \frac{\prod_{s=l}^{k-1} c_s}{\det M_{[l+1:k-1]}} \quad \text{if} \quad l \leq k-2$$

or

$$h \leq \frac{\prod_{s=k}^{l-1} b_s}{\det M_{[k+1:l-1]}} \quad \text{if} \quad l \geq k+2.$$

Proof. Once again we consider only $l \leq k-2$ and thus $k_0 = k$; the case $l \geq k+2$ can be handled in a similar manner.

If $P_{k_0}^{-1} = P_k^{-1}$ is elementwise nonnegative, then $(P_k^{-1})_{1k} \geq 0$. Using the cofactor representation for $(P_k^{-1})_{1k}$, we obtain

$$(P_k^{-1})_{1k} = \frac{1}{\det P_k} \prod_{s=1}^{k-1} c_s \left(1 - h \frac{\det M_{[l+1:k-1]}}{\prod_{s=l}^{k-1} c_s} \right) \geq 0.$$

Since $\det P_k = \det M_k + h \det M_{l-1} \prod_{t=l}^{k-1} b_t > 0$ and $c_s > 0$ for $s = 1, \dots, k-1$, it follows that

$$h \leq \frac{\prod_{s=l}^{k-1} c_s}{\det M_{[l+1:k-1]}}.$$

Now assume $h \leq \prod_{s=l}^{k-1} c_s / \det M_{[l+1:k-1]}$. Partition P_k as in (3.2) and let $\rho_k = \frac{\det P_k}{\det M_{k-1}}$. Since $\det P_k = \det M_k + h \det M_{l-1} \prod_{t=l}^{k-1} b_t > 0$, then $\rho_k > 0$. Hence the representation

$$P_k^{-1} = \begin{pmatrix} M_{k-1}^{-1} + \frac{1}{\rho_k}(M_{k-1}^{-1}\alpha_{k-1})(\beta_{k-1}^T M_{k-1}^{-1}) & \frac{1}{\rho_k} M_{k-1}^{-1}\alpha_{k-1} \\ \frac{1}{\rho_k}\beta_{k-1}^T M_{k-1}^{-1} & \frac{1}{\rho_k} \end{pmatrix}$$

implies $P_k^{-1} \geq 0$ if and only if $M_{k-1}^{-1}\alpha_{k-1} \geq 0$. We show $M_{k-1}^{-1}\alpha_{k-1} \geq 0$ by first assuming M_n is row diagonally dominant and then extending the result to a general M-matrix.

If M_n is row diagonally dominant, showing $M_{k-1}^{-1}\alpha_{k-1} \geq 0$ is equivalent to proving

$$c_{k-1}(M_{k-1}^{-1})_{i,k-1} - h(M_{k-1}^{-1})_{il} \geq 0 \tag{3.3}$$

for $i = 1, \dots, k-1$. Inequality (3.3) is established by considering two cases. Define $\det M_0 = 1$ and $\prod_{s=l}^{l-1} c_s = 1$. For $i \geq l+1$, the assumption on h, Lemma 2.3, and

Corollary 2.4 give

$$\begin{aligned}
&c_{k-1}(M_{k-1}^{-1})_{i,k-1} - h(M_{k-1}^{-1})_{il} \\
&\quad \geq c_{k-1}(M_{k-1}^{-1})_{i-1,k-1} - h(M_{k-1}^{-1})_{i-1,l} \\
&\quad \geq c_{k-1}(M_{k-1}^{-1})_{l,k-1} - h(M_{k-1}^{-1})_{ll} \\
&\quad = \frac{\det M_{l-1}}{\det M_{k-1}} \left(\prod_{s=l}^{k-1} c_s - h \det M_{[l+1:k-1]} \right) \\
&\quad \geq 0.
\end{aligned}$$

If $i \leq l$, we have

$$\begin{aligned}
&c_{k-1}(M_{k-1}^{-1})_{i,k-1} - h(M_{k-1}^{-1})_{il} \\
&\quad = \frac{1}{\det M_{k-1}} \left(\det M_{i-1} \prod_{s=i}^{k-1} c_s - h \det M_{i-1} \prod_{s=i}^{l-1} c_s \det M_{[l+1:k-1]} \right) \\
&\quad = \frac{\det M_{i-1}}{\det M_{k-1}} \prod_{s=i}^{l-1} c_s \left(\prod_{s=l}^{k-1} c_s - h \det M_{[l+1:k-1]} \right) \\
&\quad \geq 0,
\end{aligned}$$

which establishes the result.

If M_n is a general M-matrix, then, as mentioned in the introduction, there exists a positive diagonal matrix $D_n = \operatorname{diag}(d_1, \ldots, d_n)$ such that $M_n D_n$ is strictly row diagonally dominant. Define $A_n = P_n D_n = \hat{M}_n + \hat{h} E_{l,k}$, where $\hat{M}_n = M_n D_n$ and $\hat{h} = h d_k$. With the above result, given $h \leq \prod_{s=l}^{k-1} c_s / \det M_{[l+1:k-1]}$ or $\hat{h} \leq \prod_{s=l}^{k-1} \hat{c}_s / \det \hat{M}_{[l+1:k-1]}$, we have $\hat{M}_{k-1}^{-1} \hat{\alpha}_{k-1} \geq 0$. Notice that $\hat{M}_{k-1} = M_{k-1} D_{k-1}$ and $\hat{\alpha}_{k-1} = d_k \alpha_{k-1}$. Then $M_{k-1}^{-1} \alpha_{k-1} = \frac{1}{d_k} D_{k-1} \hat{M}_{k-1}^{-1} \hat{\alpha}_{k-1} \geq 0$, and the proof is complete. ∎

We now arrive at our first main result.

Theorem 3.3. *Assume M_n is a tridiagonal M-matrix. Let $P_n = M_n + hE_{l,k}$, where $h \geq 0$ and $|l - k| \geq 2$. Then P_n^{-1} is elementwise nonnegative if and only if h satisfies*

$$h \leq \frac{\prod_{s=l}^{k-1} c_s}{\det M_{[l+1:k-1]}} \quad \textit{if} \quad l \leq k-2$$

or

$$h \leq \frac{\prod_{s=k}^{l-1} b_s}{\det M_{[k+1:l-1]}} \quad \textit{if} \quad l \geq k+2.$$

Proof. The proof is an immediate consequence of Lemmas 3.1 and 3.2. ∎

We now provide an example to demonstrate Theorem 3.3. Consider perturbing a general element of the tridiagonal M-matrix

$$M = \begin{pmatrix} 8 & -6 & 0 & 0 & 0 & 0 \\ -1 & 7 & -1 & 0 & 0 & 0 \\ 0 & -2 & 6 & -1 & 0 & 0 \\ 0 & 0 & -2 & 5 & -2 & 0 \\ 0 & 0 & 0 & -1 & 4 & -3 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{pmatrix}. \tag{3.4}$$

If we perturb element $(l,k)=(1,3)$ of M by $h \geq 0$, Theorem 3.3 says the inverse of $P_1 = M + hE_{1,3}$ is nonnegative if and only if

$$h \leq \frac{\prod_{s=l}^{k-1} c_s}{\det M_{[l+1:k-1]}} = \frac{c_1 c_2}{\det M_{[2:2]}} = \frac{c_1 c_2}{a_2} = \frac{6}{7} \approx 0.857142.$$

Suppose instead we perturb element $(l,k)=(5,2)$ of M by $h \geq 0$ and define $P_2 = M + hE_{5,2}$. Then $P_2^{-1} \geq 0$ if and only if

$$h \leq \frac{\prod_{s=k}^{l-1} b_s}{\det M_{[k+1:l-1]}} = \frac{b_2 b_3 b_4}{\det M_{[3:4]}}.$$

Since $M_{[3:4]} = \left(\begin{smallmatrix} 6 & -1 \\ -2 & 5 \end{smallmatrix}\right)$, clearly we have $\det M_{[3:4]} = 28$, and hence $P_2^{-1} \geq 0$ if and only if $h \leq \frac{4}{28} = \frac{1}{7} \approx 0.142857$. The necessity of these bounds is easily verified by direct calculation.

3.2. Comments and comparisons. It is worthwhile to note some consequences of Theorem 3.3 and compare them with existing results.

Denote the maximum allowable perturbation to element (l,k) by $\delta_{l,k}$, defined by

$$\delta_{l,k} := \begin{cases} \frac{\prod_{s=l}^{k-1} c_s}{\det M_{[l+1:k-1]}} & \text{if } l \leq k-2, \\ \frac{\prod_{s=k}^{l-1} b_s}{\det M_{[k+1:l-1]}} & \text{if } l \geq k+2. \end{cases} \tag{3.5}$$

The expressions on the right, and hence the maximum allowable single element perturbation, do not depend on the dimension n of the matrix but only on the position of the perturbed element.

If M_n is a row diagonally dominant tridiagonal M-matrix for $l \leq k-2$, Corollary 2.4 ensures that

$$\begin{aligned} c_k \det M_{[l+1:k-1]} &= (M_{[l+1:k+1]}^{-1})_{k-l,k+1-l} \det M_{[l+1:k+1]} \\ &\leq (M_{[l+1:k+1]}^{-1})_{k+1-l,k+1-l} \det M_{[l+1:k+1]} \\ &= \det M_{[l+1:k]}. \end{aligned}$$

This proves $\delta_{l,k+1} \leq \delta_{l,k}$, and similarly we find $\delta_{l,k} \leq \delta_{l,k+1}$ for $k \leq l-2$. So if we increase the column index of the perturbed element above the tridiagonal band, then the actual maximum allowable perturbation decreases. Similarly, if we decrease the column index of the perturbed element below the tridiagonal band, the actual maximum allowable perturbation also decreases. Generally, the farther the perturbed element is from the main diagonal, the smaller the maximum allowable perturbation. These qualitative remarks were also noted from the sufficient bounds obtained in [13, 16, 15].

Returning now to the matrix from (3.4), direct evaluation of (3.5) gives that the maximum allowable single perturbation for elements $(1,3)$, $(1,4)$, $(1,5)$, and $(1,6)$ are (to four decimal places) $0.8571, 0.1500, 0.0645$, and 0.0542, respectively. Therefore it is clear that

$$\delta_{1,3} > \delta_{1,4} > \delta_{1,5} > \delta_{1,6}.$$

If the tridiagonal M-matrix has a Toeplitz structure, with constant diagonal values $-b, a$, and $-c$, then the actual upper bound on a single element perturbation is constant along a diagonal. Specifically,

$$\delta_{l,k} = \delta_{l+1,k+1} = \begin{cases} \frac{c^{k-l}}{\det M_{k-l-1}} & \text{if } l \le k-2, \\ \frac{b^{l-k}}{\det M_{l-k-1}} & \text{if } l \ge k+2. \end{cases}$$

Theorem 4 from [15] gives the following sufficient upper bounds on single element perturbations:

$$\begin{array}{rclcl} h & \le & c_l \prod_{s=l+1}^{k-1} \frac{c_s}{a_s} & \text{if} & l \le k-2, \\ h & \le & b_{l-1} \prod_{s=k+1}^{l-1} \frac{b_{s-1}}{a_s} & \text{if} & l \ge k+2. \end{array} \tag{3.6}$$

Compared to the actual upper bounds from Theorem 3.3, the sufficient upper bounds in (3.6) use elements from only one side of the main diagonal. When perturbing the element $(l, l+2)$ or $(l, l-2)$ of M_n, the sufficient bounds from [15] agree exactly with the actual bounds from Theorem 3.3 since

$$\frac{\prod_{s=l}^{k-1} c_s}{\det M_{[l+1:k-1]}} = \frac{c_l c_{l+1}}{a_{l+1}} \quad \text{for} \quad k = l+2$$

and

$$\frac{\prod_{s=k}^{l-1} b_s}{\det M_{[k+1:l-1]}} = \frac{b_k b_{k+1}}{a_{k+1}} \quad \text{for} \quad l = k+2.$$

However, if other elements of M_n are perturbed, the Hadamard inequality for M-matrices [14] tells us that

$$\det M_{[l+1:k-1]} < \prod_{s=l+1}^{k-1} a_s \quad \text{if} \quad l \le k-2,$$

$$\det M_{[k+1:l-1]} < \prod_{s=k+1}^{l-1} a_s \quad \text{if} \quad l \ge k+2,$$

and thus the sufficient upper bounds from (3.6) are strictly smaller than the actual upper bounds from Theorem 3.3 (and hence overly restrictive).

To give our results some additional context, we consider the work of Bouchon [3], who studies general matrix perturbations of M-matrices and gives an explicit bound on the norm of the perturbation to ensure that the perturbed irreducibly diagonally dominant M-matrix is monotone. To state Bouchon's result we introduce some notation. Let $A = (a_{ij})$ and $\tilde{A} = (\tilde{a}_{ij})$ be two real square matrices of dimension n, and define the following sets:

- $\forall (i,j) \in \{1,\dots,n\}^2$, $C_{i,j} = \{k \in \mathbb{N},\ \exists (j_0,\dots,j_k) \in \mathbb{N}^{k+1}$ such that $j_0 = i$, $j_k = j$, $a_{j_{l-1}j_l} \neq 0\ \forall l \in \{1,\dots,k\}\}$.
- $m(A) = \min_{i=1,\dots,n}(|a_{ii}|)$.
- $\eta(A) = \max_{1 \le i,j \le n, a_{ij} \neq 0} \frac{|a_{ii}|}{|a_{ij}|}$.
- $\omega = \max_{1 \le i,j \le n, \tilde{a}_{ij} \neq 0} d(i,j)$,
  where $d(i,j) = \min\{k \in C_{i,j}\}$ with $d(i,i) = 0\ \forall i$ and $d(i,j) = \infty$ if the set $C_{i,j}$ is empty.

Bouchon proves the following result.

THEOREM 3.4. *Let* $A = (a_{ij})$ *and* $\tilde{A} = (\tilde{a}_{ij})$ *be two real* $n \times n$ *matrices. Suppose that* A *is an irreducibly diagonally dominant* M*-matrix and that* $\tilde{A}\mathbf{1} \geq 0$ *with* $\mathbf{1} = (1, \dots, 1)^T \in \mathbb{R}^n$*. If* $||\tilde{A}||_\infty < Cm(A)$ *with*

$$C = \frac{1}{\eta(A)^\omega \omega e},$$

where $e = 2.7183\dots$*, then the matrix* $A + \tilde{A}$ *is monotone.*

For example, consider a single perturbation made to element (l, k) of a row diagonally dominant tridiagonal Toeplitz M-matrix M_n, i.e., $M_n = \text{tridiag}(-b, a, -c)$. It is easy to find $\omega = |k - l|$, $\eta(M_n) = \frac{a}{\min\{b,c\}}$, and $m(M_n) = a$. Thus, from Theorem 3.4, if

$$h < \frac{1}{|k-l|e} \cdot \frac{(\min\{b,c\})^{|k-l|}}{a^{|k-l|-1}}, \tag{3.7}$$

then $(M_n + hE_{l,k})^{-1}$ is elementwise nonnegative. As in Theorem 3.3, the upper bound on the perturbation h provided by Theorem 3.4 is independent of the matrix dimension n. Furthermore, the bound decreases as the perturbed element is moved farther from the diagonal.

As an example, consider perturbing a single element (l, k) of the tridiagonal M-matrix

$$M = \begin{pmatrix} 4 & -2 & 0 & 0 & 0 \\ -1 & 4 & -2 & 0 & 0 \\ 0 & -1 & 4 & -2 & 0 \\ 0 & 0 & -1 & 4 & -2 \\ 0 & 0 & 0 & -1 & 4 \end{pmatrix}. \tag{3.8}$$

When perturbing element $(l, k) = (1, 3)$ of M by h, inequality (3.7) says that if

$$h < \frac{1}{|k-l|e} \cdot \frac{(\min\{b,c\})^{|k-l|}}{a^{|k-l|-1}} = \frac{1}{2e} \cdot \frac{1}{4} \approx 0.04598,$$

then $(M + hE_{1,3})^{-1}$ is elementwise positive. However, from Theorem 3.3, the inverse of $M + hE_{1,3}$ is nonnegative if and only if

$$h \leq \frac{c^{k-l}}{\det M_{k-l-1}} = \frac{c^2}{a} = \frac{2^2}{4} = 1.$$

If element $(l, k) = (5, 2)$ of M is perturbed by a value h, then our result indicates $(M + hE_{5,2})^{-1} \geq 0$ if and only if

$$h \leq \frac{b^{l-k}}{\det M_{l-k-1}} = \frac{b^3}{\det M_2} = \frac{1}{\det\begin{pmatrix} 4 & -2 \\ -1 & 4 \end{pmatrix}} = \frac{1}{14} \approx 0.0714286.$$

Bouchon's theorem requires

$$h < \frac{1}{|k-l|e} \cdot \frac{\min\{b,c\}^{|k-l|}}{a^{|k-l|-1}} = \frac{1}{3e} \cdot \frac{1}{4^2} = \frac{1}{48e} \approx 0.00766.$$

Clearly the sufficient normwise bound of Bouchon is not as sharp as the elementwise bound provided by Theorem 3.3. This is not unexpected, as Bouchon's result is much more general, allowing arbitrary matrix perturbations.

TABLE 3.1
Maximum allowable perturbations of M in (3.8).

Position	(1,3)	(1,4)	(1,5)	(3,1)	(4,1)	(5,1)
Bouchon	0.04598	0.00766	0.00143	0.04598	0.00766	0.00143
Old bound	1.00000	0.50000	0.25000	0.25000	0.06250	0.01560
New bound	1.00000	0.57143	0.33333	0.25000	0.07143	0.02083
*	1.00000	0.57143	0.33333	0.25000	0.07143	0.02083

In Table 3.1, we give maximum allowable perturbations for the matrix (3.8). We present the sufficient upper bounds from (3.7) (Bouchon), from (3.6) (old bound), the necessary and sufficient bounds provided by Theorem 3.3 (new bound), and the actual bounds, computed numerically in [16] (*). For this example, Table 3.1 generally shows that the sufficient upper bounds provided by Bouchon are overly conservative. Also, we see that our new bounds from Theorem 3.3 agree with the values computed numerically in [16] and are improvements over those found in [15].

4. Simultaneous perturbations of many elements. In this section, we consider the effects of multiple simultaneous elementwise perturbations on the inverses of tridiagonal M-matrices. Let M_n be a tridiagonal M-matrix, and let

$$P_n = M_n + \sum_{(l,k)\in\mathcal{I}} h_{l,k}E_{l,k}, \tag{4.1}$$

where $h_{l,k} \geq 0$, and $\mathcal{I}$ is a subset of all possible ordered pairs (l,k), where $|k-l| \geq 2$ and $l,k = 1,\dots,n$. We provide necessary and sufficient conditions to ensure $P_n^{-1} \geq 0$.

To begin we prove the following lemma.

LEMMA 4.1. *Let M_n be a tridiagonal M-matrix. Let $P_n = M_n + \sum_{(l,k)\in\mathcal{I}} h_{l,k}E_{l,k}$, $\alpha_{n-1} = (-h_{1,n},\dots,-h_{n-2,n},c_{n-1})^T$, and $\beta_{n-1} = (-h_{n,1},\dots,-h_{n,n-2},b_{n-1})^T$, where $h_{l,k} \geq 0$. If $P_n^{-1} \geq 0$, then $P_{n-1}^{-1}\alpha_{n-1} \geq 0$, $\beta_{n-1}^T P_{n-1}^{-1} \geq 0$, and $P_{n-1}^{-1} \geq 0$.*

Proof. We first show that the result holds if M_n is a row and column diagonally dominant tridiagonal M-matrix. If $P_n^{-1} \geq 0$, we begin by showing $(P_n^{-1})_{j+1,n} \geq (P_n^{-1})_{jn}$ for $j = 1,\dots,n-1$ by induction. Clearly we have

$$0 = a_1(P_n^{-1})_{1n} - c_1(P_n^{-1})_{2n} + \sum_{s=3}^{n} h_{1,s}(P_n^{-1})_{sn} \geq a_1(P_n^{-1})_{1n} - c_1(P_n^{-1})_{2n},$$

that is,

$$(P_n^{-1})_{2n} \geq \frac{a_1}{c_1}(P_n^{-1})_{1n} \geq (P_n^{-1})_{1n}.$$

We now show that the condition $(P_n^{-1})_{jn} \geq (P_n^{-1})_{j-1,n}$ implies $(P_n^{-1})_{j+1,n} \geq (P_n^{-1})_{jn}$. It is easy to see that

$$\begin{aligned} 0 &= \sum_{s\neq j,j\pm1}^{n} h_{j,s}(P_n^{-1})_{sn} - b_{j-1}(P_n^{-1})_{j-1,n} + a_j(P_n^{-1})_{jn} - c_j(P_n^{-1})_{j+1,n} \\ &\geq -b_{j-1}(P_n^{-1})_{j-1,n} + a_j(P_n^{-1})_{jn} - c_j(P_n^{-1})_{j+1,n} \\ &\geq -b_{j-1}(P_n^{-1})_{jn} + a_j(P_n^{-1})_{jn} - c_j(P_n^{-1})_{j+1,n}, \end{aligned}$$

from which we have

$$(P_n^{-1})_{j+1,n} \geq \frac{a_j - b_{j-1}}{c_j}(P_n^{-1})_{jn} \geq (P_n^{-1})_{jn},$$

which is the desired result. Using a similar technique, we have

$$\begin{aligned} &(P_n^{-1})_{ij} \geq (P_n^{-1})_{i-1,j}, \quad (P_n^{-1})_{ij} \geq (P_n^{-1})_{i,j+1} \quad \text{for} \quad i \leq j, \\ &(P_n^{-1})_{ij} \geq (P_n^{-1})_{i+1,j}, \quad (P_n^{-1})_{ij} \geq (P_n^{-1})_{i,j-1} \quad \text{for} \quad i \geq j. \end{aligned} \tag{4.2}$$

Using this information, we now argue by contradiction that $\det P_{n-1} \neq 0$. Suppose $\det P_{n-1} = 0$; then $(P_n^{-1})_{nn} = \det P_{n-1}/\det P_n = 0$. It follows that $(P_n^{-1})_{n-1,n} \leq (P_n^{-1})_{nn} = 0$ and thus $(P_n^{-1})_{n-1,n} = 0$. Recursively, we have $(P_n^{-1})_{1n} = (P_n^{-1})_{2n} = \cdots = (P_n^{-1})_{nn} = 0$, which violates the nonsingularity of P_n.

To complete the proof, define $\rho_n = \frac{\det P_n}{\det P_{n-1}}$ and partition P_n as

$$P_n = \begin{pmatrix} P_{n-1} & -\alpha_{n-1} \\ -\beta_{n-1}^T & a_n \end{pmatrix}. \tag{4.3}$$

The inverse may be written as

$$P_n^{-1} = \begin{pmatrix} P_{n-1}^{-1} + \frac{1}{\rho_n}(P_{n-1}^{-1}\alpha_{n-1})(\beta_{n-1}^T P_{n-1}^{-1}) & \frac{1}{\rho_n} P_{n-1}^{-1}\alpha_{n-1} \\ \frac{1}{\rho_n}\beta_{n-1}^T P_{n-1}^{-1} & \frac{1}{\rho_n} \end{pmatrix}. \tag{4.4}$$

The assumption $P_n^{-1} \geq 0$ requires $\rho_n = \frac{1}{(P_n^{-1})_{nn}} > 0$, $P_{n-1}^{-1}\alpha_{n-1} \geq 0$, and $\beta_{n-1}^T P_{n-1}^{-1} \geq 0$. Furthermore, each element of the upper left block of P_n^{-1} has the form

$$(P_n^{-1})_{ij} = (P_{n-1}^{-1})_{ij} + \rho_n (P_n^{-1})_{in}(P_n^{-1})_{nj}$$

or

$$(P_{n-1}^{-1})_{ij} = (P_n^{-1})_{ij} - \rho_n (P_n^{-1})_{in}(P_n^{-1})_{nj}$$

for $i, j = 1, \ldots, n-1$. In fact, from (4.2), we have for $i \geq j$,

$$\begin{aligned} (P_{n-1}^{-1})_{ij} &= (P_n^{-1})_{ij} - \rho_n (P_n^{-1})_{in}(P_n^{-1})_{nj} \\ &\geq (P_n^{-1})_{ij} - \rho_n (P_n^{-1})_{nn}(P_n^{-1})_{nj} = (P_n^{-1})_{ij} - (P_n^{-1})_{nj} \geq 0, \end{aligned}$$

and for $i \leq j$,

$$(P_{n-1}^{-1})_{ij} \geq (P_n^{-1})_{ij} - \rho_n (P_n^{-1})_{in}(P_n^{-1})_{nn} = (P_n^{-1})_{ij} - (P_n^{-1})_{in} \geq 0.$$

Thus, $(P_{n-1}^{-1})_{ij} \geq 0$ for $i, j = 1, \ldots, n-1$ or $P_{n-1}^{-1} \geq 0$.

We now show that the result is true if M_n is a general tridiagonal M-matrix. Recall from Lemma 2.2 that there exist two positive diagonal matrices $X_n = \mathrm{diag}(x_1, \ldots, x_n)$ and $Y_n = \mathrm{diag}(y_1, \ldots, y_n)$ such that $Y_n M_n X_n$ is strictly diagonally dominant. Let $B_n = Y_n P_n X_n = \hat{M}_n + \sum_{(l,k)\in\mathcal{I}} \hat{h}_{l,k} E_{l,k}$, where $\hat{M}_n = Y_n M_n X_n$ and $\hat{h}_{l,k} = h_{l,k} x_k y_l$. Thus, if $P_n^{-1} \geq 0$, then $B_n^{-1} = X_n^{-1} P_n^{-1} Y_n^{-1} \geq 0$. According to the result proven above, we have $B_{n-1}^{-1} \geq 0$, $B_{n-1}^{-1}\hat{\alpha}_{n-1} \geq 0$, and $\hat{\beta}_{n-1}^T B_{n-1}^{-1} \geq 0$, where $\hat{\alpha}_{n-1} = Y_{n-1}\alpha_{n-1} x_n$ and $\hat{\beta}_{n-1} = X_{n-1}\beta_{n-1} y_n$. As a result, the inequalities $P_{n-1}^{-1} = X_{n-1} B_{n-1}^{-1} Y_{n-1} \geq 0$, $P_{n-1}^{-1}\alpha_{n-1} = \frac{1}{x_n} X_{n-1} B_{n-1}^{-1}\hat{\alpha}_{n-1} \geq 0$, and $\beta_{n-1}^T P_{n-1}^{-1} = \frac{1}{y_n}\hat{\beta}_{n-1}^T B_{n-1}^{-1} Y_{n-1} \geq 0$ hold, giving the desired result. $\square$

Lemma 4.1 now allows us to obtain the following necessary and sufficient condition preserving monotonicity of P_n.

THEOREM 4.2. *Let M_n be a tridiagonal M-matrix. Let $P_n = M_n + \sum_{(l,k)\in\mathcal{I}} h_{l,k}E_{l,k}$, $\alpha_{n-1} = (-h_{1,n}, \ldots, -h_{n-2,n}, c_{n-1})^T$, and $\beta_{n-1} = (-h_{n,1}, \ldots, -h_{n,n-2}, b_{n-1})^T$, where $h_{l,k} \geq 0$. Then $P_n^{-1} \geq 0$ if and only if $P_{n-1}^{-1} \geq 0$, $P_{n-1}^{-1}\alpha_{n-1} \geq 0$, and $\beta_{n-1}^T P_{n-1}^{-1} \geq 0$.*

Proof. Lemma 4.1 proves that if $P_n^{-1} \geq 0$, then $P_{n-1}^{-1}\alpha_{n-1} \geq 0, \beta_{n-1}^T P_{n-1}^{-1} \geq 0$, and $P_{n-1}^{-1} \geq 0$. We now consider the proof in the other direction.

Assume $P_{n-1}^{-1} \geq 0$. We know both $M_{n-1}^{-1} \geq 0$ and $P_{n-1} \geq M_{n-1}$; thus we have $0 \leq P_{n-1}^{-1} \leq M_{n-1}^{-1}$, and hence $(P_{n-1}^{-1})_{n-1,n-1} \leq (M_{n-1}^{-1})_{n-1,n-1}$. Since by assumption, $P_{n-1}^{-1}\alpha_{n-1} \geq 0$ and $\beta_{n-1}^T P_{n-1}^{-1} \geq 0$, then defining $\rho_n = \frac{\det P_n}{\det P_{n-1}}$ we have

$$\begin{aligned}\rho_n = a_n - \beta_{n-1}^T P_{n-1}^{-1}\alpha_{n-1} = a_n + (-\beta_{n-1}^T P_{n-1}^{-1})\alpha_{n-1} &\geq a_n - b_{n-1}(P_{n-1}^{-1})_{n-1,n-1}c_{n-1} \\ &\geq a_n - b_{n-1}(M_{n-1}^{-1})_{n-1,n-1}c_{n-1} = \frac{\det M_n}{\det M_{n-1}} > 0.\end{aligned}$$

Hence, partitioning P_n as in (4.3), the requirement $P_n^{-1} \geq 0$ follows from (4.4). ◻

Theorem 4.2 replaces the question of monotonicity of the n-by-n matrix P_n with a decision concerning the nonnegativity of P_{n-1}^{-1}, $P_{n-1}^{-1}\alpha_{n-1}$, and $\beta_{n-1}^T P_{n-1}^{-1}$, involving a matrix of size $n-1$-by-$n-1$. Also, $P_{n-1}^{-1} \geq 0$ if and only if $P_{n-2}^{-1} \geq 0$, $P_{n-2}^{-1}\alpha_{n-2} \geq 0$, and $\beta_{n-2}^T P_{n-2}^{-1} \geq 0$. Recursively we have $P_3^{-1} \geq 0$ if and only if $P_2^{-1} \geq 0$, $P_2^{-1}\alpha_2 \geq 0$, and $\beta_2^T P_2^{-1} \geq 0$. This gives the following corollary.

COROLLARY 4.3. *Let M_n and P_n be as defined in Theorem 4.2. Define the vectors $\alpha_s = (-h_{1,s+1}, \ldots, -h_{s-1,s+1}, c_s)^T$ and $\beta_s = (-h_{s+1,1}, \ldots, -h_{s+1,s-1}, b_s)^T$ for $s = 2, \ldots, n-1$. Then $P_n^{-1} \geq 0$ if and only if $P_s^{-1}\alpha_s \geq 0$ and $\beta_s^T P_s^{-1} \geq 0$ for $s = 2, \ldots, n-1$.*

Theorem 4.2 describes an iteratively defined, necessary, and sufficient condition so that the perturbed matrix P_n from (4.1) is monotone. To demonstrate this result we first consider Theorem 4.2 for the special case of a single element perturbation. In this case (4.1) becomes

$$P_n = M_n + hE_{l,k} \quad \text{for } |l-k| \geq 2.$$

Without loss of generality, we consider only $l \leq k-2$. Here $\alpha_s = (0, \ldots, 0, c_s)^T \geq 0$ and $\beta_s = (0, \ldots, 0, b_s)^T \geq 0$ for $s = k, \ldots, n-1$. From Theorem 4.2, we have $P_n^{-1} \geq 0$ if and only if $P_{n-1}^{-1} \geq 0$, since $\alpha_{n-1} \geq 0$ and $\beta_{n-1} \geq 0$. Similarly, $P_{n-1}^{-1} \geq 0$ if and only if $P_{n-2}^{-1} \geq 0$, since $\alpha_{n-2} \geq 0$ and $\beta_{n-2} \geq 0$. Moreover, $P_{s+1}^{-1} \geq 0$ if and only if $P_s^{-1} \geq 0$ for $s = k, \ldots, n-3$. Thus we have $P_n^{-1} \geq 0$ if and only if $P_k^{-1} \geq 0$. This is exactly the content of Lemma 3.1. Recall that $P_{k-1}^{-1} = M_{k-1}^{-1} \geq 0$ and $\beta_{k-1} = (0, \ldots, 0, b_{k-1})^T \geq 0$. Therefore $P_k^{-1} \geq 0$ if and only if $M_{k-1}^{-1}\alpha_{k-1} \geq 0$, which is equivalent to the requirement

$$h \leq \frac{\prod_{s=l}^{k-1} c_s}{\det M_{[l+1:k-1]}}.$$

This result is given in Lemma 3.2. Thus, Theorem 3.3 is a special case of Theorem 4.2.

As a second example, consider the perturbed tridiagonal M-matrix

$$P_4 = \begin{pmatrix} a_1 & -c_1 & h_1 & 0 \\ -b_1 & a_2 & -c_2 & h_2 \\ 0 & -b_2 & a_3 & -c_3 \\ 0 & 0 & -b_3 & a_4 \end{pmatrix}.$$

From Theorem 3.3, we have $P_3^{-1} \geq 0$ if and only if $h_1 \leq c_1c_2/a_2$. Given $h_1 \leq c_1c_2/a_2$, it is clear that $\beta_3^T P_3^{-1} \geq 0$ since $\beta_3 = (0, 0, b_3)^T \geq 0$. Thus, Theorem 4.2 ensures that $P_4^{-1} \geq 0$ if and only if $h_1 \leq c_1c_2/a_2$ and $P_3^{-1}\alpha_3 \geq 0$. A simple calculation shows that, given $h_1 \leq c_1c_2/a_2$, $P_3^{-1}\alpha_3 \geq 0$ if and only if $h_2 \leq c_3(P_3^{-1})_{i,3}/(P_3^{-1})_{i,2}$ for $i = 1, 2, 3$. To compute $\min_{i=1,2,3}(P_3^{-1})_{i,3}/(P_3^{-1})_{i,2}$, we rewrite P_3 as $P_3 = M_3 + h_1E_{1,3}$. Recall from section 1 that there exists a positive diagonal matrix $D_3 = \operatorname{diag}(d_1, d_2, d_3)$ such that $\hat{M}_3 := M_3D_3$ is strictly row diagonally dominant. Define

$$\hat{P}_3 = P_3D_3 = \begin{pmatrix} \hat{a}_1 & -\hat{c}_1 & \hat{h}_1 \\ -\hat{b}_1 & \hat{a}_2 & -\hat{c}_2 \\ 0 & -\hat{b}_2 & \hat{a}_3 \end{pmatrix};$$

then $h_1 \leq c_1c_2/a_2$ ensures $\hat{h}_1 = h_1d_3 \leq (c_1d_2)(c_2d_3)/(a_2d_2) = \hat{c}_1\hat{c}_2/\hat{a}_2$. Using the fact that $\hat{M}_3$ is strictly row diagonally dominant and the upper bound on $\hat{h}_1$ gives

$$\frac{(\hat{P}_3^{-1})_{1,3}}{(\hat{P}_3^{-1})_{1,2}} = \frac{\hat{c}_1\hat{c}_2 - \hat{a}_2\hat{h}_1}{\hat{a}_3\hat{c}_1 - \hat{h}_1\hat{b}_2} < \frac{(\hat{P}_3^{-1})_{2,3}}{(\hat{P}_3^{-1})_{2,2}} = \frac{\hat{a}_1\hat{c}_2 - \hat{b}_1\hat{h}_1}{\hat{a}_1\hat{a}_3} < \frac{(\hat{P}_3^{-1})_{3,3}}{(\hat{P}_3^{-1})_{3,2}} = \frac{\hat{a}_1\hat{a}_2 - \hat{b}_1\hat{c}_1}{\hat{a}_1\hat{b}_2}.$$

Hence the relation $P_3^{-1} = D_3\hat{P}_3^{-1}$ implies

$$\min_{i=1,2,3} \frac{(P_3^{-1})_{i,3}}{(P_3^{-1})_{i,2}} = \min_{i=1,2,3} \frac{d_i(\hat{P}_3^{-1})_{i,3}}{d_i(\hat{P}_3^{-1})_{i,2}} = \frac{(\hat{P}_3^{-1})_{1,3}}{(\hat{P}_3^{-1})_{1,2}} = \frac{(P_3^{-1})_{1,3}}{(P_3^{-1})_{1,2}} = \frac{c_1c_2 - h_1a_2}{a_3c_1 - h_1b_2}.$$

Thus, we require

$$h_2 \leq \frac{c_1c_2c_3 - h_1a_2c_3}{a_3c_1 - h_1b_2}.$$

Therefore, $P_4^{-1} \geq 0$ if and only if

$$h_1 \leq \frac{c_1c_2}{a_2} \quad \text{and} \quad h_2 \leq \frac{c_1c_2c_3 - h_1a_2c_3}{a_3c_1 - h_1b_2}.$$

Particularly, if $h_2 = h_1$, we have the requirement

$$h_1 \leq \frac{c_1c_2}{a_2} \quad \text{and} \quad h_1 \leq \frac{c_1c_2c_3 - h_1a_3c_1}{a_2c_3 - h_1b_2}.$$

In comparison, for a two element perturbation in positions (l_1, k_1) and (l_2, k_2), Bouchon's Theorem 3.4 gives the following easily computable bounds on h_1 and h_2 to ensure monotonicity:

$$\begin{aligned} &\max\{h_1, h_2\} < \frac{\min\{a_i\}}{\omega e \cdot \max\{\frac{a_i}{c_i}, \frac{a_i}{b_{i-1}}\}^\omega} && \text{if } l_1 \neq l_2, \\ &h_1 + h_2 < \frac{\min\{a_i\}}{\omega e \cdot \max\{\frac{a_i}{c_i}, \frac{a_i}{b_{i-1}}\}^\omega} && \text{if } l_1 = l_2, \end{aligned} \tag{4.5}$$

with $\omega = \max\{|k_1 - l_1|, |k_2 - l_2|\}$.

To compare the two results, we consider the following specific example. Let

$$P_4' = \begin{pmatrix} 10 & -1 & h_1 & 0 \\ -2 & 50 & -8 & h_2 \\ 0 & -8 & 100 & -2 \\ 0 & 0 & -1 & 20 \end{pmatrix}.$$

For $h = h_1 = h_2$, (4.5) requires $h < \frac{10}{2e\cdot 50^2} = \frac{1}{500e} \approx 7.3576 \cdot 10^{-4}$. However, from Theorem 4.2, P_4' is monotone if and only if

$$h \le \frac{c_1c_2}{a_2} = 0.16 \quad \text{and} \quad h \le \frac{c_1c_2c_3 - ha_3c_1}{a_2c_3 - hb_2} = \frac{16 - 100h}{100 - 8h}$$

or $h \le 0.080257$. In fact, the inverse of P_4' with $h = 0.080257$ is

$$P_4'^{-1} = \begin{pmatrix} 0.1004 & 0.0020 & 0.0001 & 0.0000 \\ 0.0041 & 0.0203 & 0.0016 & 0.0001 \\ 0.0003 & 0.0016 & 0.0101 & 0.0010 \\ 0.0000 & 0.0001 & 0.0005 & 0.0501 \end{pmatrix}.$$

Indeed, our result gives a tight bound on the maximum allowable perturbation and a large improvement over the result of Bouchon.

We now consider the monotonicity of P_n of the form

$$P_n = M_n + vE_n, \tag{4.6}$$

where $v \ge 0$ and $E_n = (e_{ij})$ is a nonnegative matrix with $e_{ij} = 0$ if $|i-j| \le 1$. Corollary 4.3 allows us to construct an algorithm, which we state as Theorem 4.4, to find the necessary and sufficient upper bound on v to ensure monotonicity of the perturbed matrix P_n in (4.6).

THEOREM 4.4. *Let M_n be a tridiagonal M-matrix, let $P_n = M_n + vE_n$ where $v \ge 0$, and let $E_n = (e_{ij}) \ge 0$ with $e_{ij} = 0$ if $|i-j| < 2$. Let $v_1 = \infty$ and $v_s \in [0, v_{s-1}]$ be the largest, possibly infinite, value of v such that $P_s^{-1}\alpha_s \ge 0$ and $\beta_s^T P_s^{-1} \ge 0$, where $\alpha_s = (-ve_{1,s+1}, \dots, -ve_{s-1,s+1}, c_s)^T$ and $\beta_s = (-ve_{s+1,1}, \dots, -ve_{s+1,s-1}, b_s)^T$ for $s = 2, \dots, n-1$. Then $P_n^{-1} \ge 0$ if and only if $v \le v_{n-1}$.*

Proof. We first show that if $P_n^{-1} \ge 0$, then $v \le v_{n-1}$. Corollary 4.3 says that if $P_n^{-1} \ge 0$, then $P_s^{-1}\alpha_s \ge 0$ and $\beta_s^T P_s^{-1} \ge 0$ for $s = 2, \dots, n-1$. Given $P_2^{-1}\alpha_2 \ge 0$ and $\beta_2^T P_2^{-1} \ge 0$, the definition of v_2 gives $v \le v_2$. Recall that $v_3 \in [0, v_2]$ is the largest value such that $P_3^{-1}\alpha_3 \ge 0$ and $\beta_3^T P_3^{-1} \ge 0$, thus we have $v \le v_3$. Recursively, we must have $v \le v_{n-1}$.

We now show that if $v \le v_{n-1}$, then $P_n^{-1} \ge 0$. For $s \le n-1$, $v_s \in [0, v_{s-1}]$ implies $v \le v_{n-1} \le v_{n-2} \le \cdots \le v_3 \le v_2$. From the definition of v_2, $v \le v_2$ gives $P_2^{-1}\alpha_2 \ge 0$ and $\beta_2^T P_2^{-1} \ge 0$. Also $v \le v_3 \le v_2$ gives $P_3^{-1}\alpha_3 \ge 0$ and $\beta_3^T P_3^{-1} \ge 0$. Recursively, $v \le v_{n-1} \le \cdots \le v_2$ implies $P_{n-1}^{-1}\alpha_{n-1} \ge 0$ and $\beta_{n-1}^T P_{n-1}^{-1} \ge 0$. Thus, we have $P_s^{-1}\alpha_s \ge 0$ and $\beta_s^T P_s^{-1} \ge 0$ for $s = 2, \dots, n-1$. Corollary 4.3 then gives $P_n^{-1} \ge 0$. ∎

It is useful to note that the vectors α_s and β_s in Theorem 4.4 may be extracted from P_n (in (4.1) and (4.6)) as $\alpha_s = -P_n(1:s, s+1)$ and $\beta_s = -P_n(s+1, 1:s)^T$ for $s = 2, \dots, n-1$.

For our algorithm, we iteratively construct v_s for $s = 2, \dots, n-1$ according to Theorem 4.4. Particularly, at iteration s, we already have $v \le v_{s-1} \le v_{s-2} \le \cdots \le v_2$, which ensures that $P_j^{-1} \ge 0$, implying $\det P_j > 0$ for $j = 3, \dots, s$. Thus defining $\rho_j = \frac{\det P_j}{\det P_{j-1}}$, we have

$$P_j^{-1} = \begin{pmatrix} P_{j-1}^{-1} + \frac{1}{\rho_j}(P_{j-1}^{-1}\alpha_{j-1})(\beta_{j-1}^T P_{j-1}^{-1}) & \frac{1}{\rho_j}P_{j-1}^{-1}\alpha_{j-1} \\ \frac{1}{\rho_j}\beta_{j-1}^T P_{j-1}^{-1} & \frac{1}{\rho_j} \end{pmatrix}$$

for $j = 2, \dots, s$. With this special partitioning, we recursively construct P_s^{-1} only using matrix-vector multiplications. In our experience, this strategy is faster and more stable than using the MATLAB command *inv* to compute P_s^{-1} at each iteration.

We finally remark that if for any $s \in \{2, 3, \dots, n-1\}$ we have $P_n^{-1}(v_s)$ nonnegative, then Theorem 4.4 requires $v_s \le v_{n-1}$. Yet the ordering $v_{n-1} \le v_{n-2} \le \cdots \le v_3 \le v_2$ forces $v_{n-1} \le v_s$. Hence, $v_s = v_{n-1}$, and we may stop our algorithm. This provides an early, albeit expensive, termination condition for our algorithm.

Buffoni [6] presents the following algorithm to compute the largest, possibly infinite, allowable perturbation for persistence of monotonicity of a real n-by-n matrix $Z(v) = M + vE$, where M is a nonsingular matrix with positive inverse, E $(E \ge 0)$ is a nonnegative matrix, and v a nonnegative real parameter.

THEOREM 4.5. *Let v^* be the largest, possibly infinite, number such that $Z(v) > 0$ in $[0, v^*)$. Then v^* is the limit of the sequence $\{v_k\}$ given by $v_0 = 0$,*

$$v_{k+1} = v_k + \min_{i,j;\omega_{kij}>0} z_{kij}/\omega_{kij}, \quad k = 0, 1, 2, \dots, \tag{4.7}$$

where $Z_k = Z(v_k) = (M + v_k E)^{-1} = [z_{kij}]$, $W_k = -Z'(v_k) = Z_k E Z_k = [\omega_{kij}]$.

We remark that the algorithm contained in Theorem 4.5 is more general than our algorithm described in Theorem 4.4. Theorem 4.5 allows general nonnegative perturbations of inverse positive matrices. Although Theorems 4.4 and 4.5 both describe algorithms to ensure that (4.6) is monotone, each uses a different technique to compute the actual allowable upper bounds. The main differences are as follows:

1. Buffoni's algorithm involves computing the inverse of $P_n(v_{k-1})$, a matrix of size n-by-n during each iteration; the computation cost is therefore $O(n^3)$ per iteration. In contrast, at iteration k, Theorem 4.4 mainly requires a number of inverses of P_k, a matrix with dimension k. In general, a search procedure may be used to find the maximum value of v so that $P_k^{-1}\alpha_k \ge 0$ and $\beta_k^T P_k^{-1} \ge 0$. The cost of each iteration, as dictated by the number of matrix inversions required, is determined by the actual search procedure used and the accuracy needed for the upper bound. In our experience, Buffoni's algorithm is more efficient for small n or if high precision is required. However, for large matrix sizes or if a rough estimate (i.e., a sufficient bound) is needed, then our algorithm is often faster.
2. The sequence $\{v_k\}$ generated by Theorem 4.4 is nonincreasing, and the actual bound on v is obtained after at most $n - 1$ iterations. However, the actual bound v^* by Theorem 4.5 is the limit of the sequence $\{v_k\}$ of (4.7), which implies that Theorem 4.5 may fail to provide the actual bound v^* within n iterations.

As an example, we compare Theorems 3.4, 4.4, and 4.5 for the n-by-n matrix

$$P_n = \begin{pmatrix} 4 & -1 & h & & & & \\ -1 & 4 & -1 & h & & & \\ h & -1 & 4 & -1 & h & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & h & -1 & 4 & -1 & h \\ & & & h & -1 & 4 & -1 \\ & & & & h & -1 & 4 \end{pmatrix} \tag{4.8}$$

considered in [13]. It is clear that, in Bouchon's notation, $\eta(M_n) = 4$, $\omega = 2$; thus we have $C = \frac{1}{\eta(M_n)^\omega \omega e} = \frac{1}{32e}$. Moreover, we have $||E_n||_\infty = 2h$ and $m(M_n) = 4$. Therefore, Theorem 3.4 says that if $||E_n||_\infty < Cm(M_n)$, that is, $h < \frac{1}{16e} \approx 0.0230$, then P_n is monotone.

TABLE 4.1
Comparison of the maximum allowable perturbations on h for monotonicity of P_n given by (4.8).

n	50	100	200	500
Bouchon	0.0230	0.0230	0.0230	0.0230
Buffoni	0.0648(31)	0.0646(64)	0.0646(132)	0.0339(87)
Our method	0.0648(44)	0.0646(73)	0.0646(73)	0.0645(207)
*	0.0648	0.0646	0.0646	0.0645

Table 4.1 lists the computed bounds on $h(=v)$ obtained by Theorems 4.4 and 4.5 to maintain the inverse positivity of P_n for different values of n. The numbers in parentheses indicate the number of iterations required by each algorithm. Moreover, the actual largest allowable value of v for which the matrix is monotone, obtained by an exhaustive search, is recorded in row *.

From Table 4.1, we see that for smaller n, Buffoni's algorithm can provide precise bounds on v with less iterations. However, for $n = 500$, the bound found by Buffoni is smaller than the one found by Theorem 4.4 (and hence only sufficient).

5. An application. In this section, we provide a practical application involving perturbations of tridiagonal M-matrices and subsequently show how our results provide bounds to ensure monotonicity.

This application was considered by Buffoni in [6]. The integro-differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[p\frac{\partial u}{\partial x}\right] + q\,[u_0 - u] + v\int_0^1 K(x,x')[u_0(x') - u(x')]dx', \tag{5.1}$$

where $p(x) > 0$, $q(x) \geq 0$, $u_0(x) \geq 0$, and $K(x,x') \geq 0$, with boundary conditions $u(0,t) = u(1,t) = 0$, is used to model population in a spatially distributed community; see [26] for details.

Steady state solutions, u_∞, of (5.1) are obtained numerically using centered differences and the trapezium rule to approximate the integral term. For $p = 1$, $q = 0$, and $K(x,x') = \exp(-(x-x')^2)$, the discrete approximation, $\mathbf{u}_\infty$, to the steady state solution satisfies

$$(M_n + vE_n)\mathbf{u}_\infty = vE_n\mathbf{u}_\mathbf{0},$$

where

$$M_n = \frac{1}{h^2}\begin{pmatrix} 2 & -1 & & \\ -1 & 2 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix},$$

$$E_n = h\begin{pmatrix} 1 & \exp(-h^2) & \cdots & \exp(-(n-1)^2h^2) \\ \exp(-h^2) & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \exp(-h^2) \\ \exp(-(n-1)^2h^2) & \cdots & \exp(-h^2) & 1 \end{pmatrix},$$

$\mathbf{u}_{\mathbf{0}j} = u_0(x_j), \mathbf{u}_{\infty j} = u_\infty(x_j)$, and $x_j = jh$ for $j = 1,\ldots,n$. The quantity $h = 1/m$ is the mesh spacing, and the matrices are n-by-n where $n = m - 1$. The inverse positivity of $M_n + vE_n$ ensures the positivity of $\mathbf{u}_\infty$.

Table 5.1
Maximum allowable perturbations on v for application 1.

m	30	40	50
Bouchon	9.8277e-8	1.2481e-10	1.5011e-13
Buffoni	8.8418(4)	8.6532(4)	8.5435(4)
Our method	8.8418(28)	8.6532(38)	8.5435(48)
*	8.8418	8.6532	8.5435

Table 5.1 lists the results of the actual upper bounds on v obtained from Theorems 4.4, 4.5, and 3.4 for $m = 30$, 40, and 50. To compute the results from Bouchon's theorem we note $\eta(M_n) = 2$, $m(M_n) = \frac{2}{h^2}$, and

$$\omega = \max_{1\le i,j\le n,(E_n)_{ij}\neq 0} d(i,j) = \max_{|l-k|\ge 2} |k-l| = n-1.$$

Monotonicity of $M_n + vE_n$ then requires

$$\begin{aligned} ||vE_n||_\infty = vh\left(1+2\sum_{j=1}^{\frac{n-1}{2}}\exp(-j^2h^2)\right) < Cm(M_n) &= \frac{1}{\eta(M_n)^\omega \omega e} m(M_n) \\ &= \frac{1}{2^{n-1}\cdot(n-1)e}\cdot\frac{2}{h^2} \end{aligned}$$

or

$$v < \frac{m^3}{\left(1+2\sum_{j=1}^{\frac{n-1}{2}}\exp(-j^2h^2)\right)\cdot 2^{n-2}\cdot(n-1)e}.$$

Also, to compute the results from Theorem 4.4, we rewrite $M_n + vE_n = \hat{M}_n + v\hat{E}_n$, where $\hat{M}_n = M_n + v(E_n - \hat{E}_n)$ and

$$\hat{E}_n = h\begin{pmatrix} 0 & 0 & \exp(-4h^2) & \dots & \exp(-(n-1)^2h^2) \\ 0 & 0 & 0 & \ddots & \vdots \\ \exp(-4h^2) & 0 & \ddots & \ddots & \exp(-4h^2) \\ \vdots & \ddots & \ddots & 0 & 0 \\ \exp(-(n-1)^2h^2) & \dots & \exp(-4h^2) & 0 & 0 \end{pmatrix}.$$

It is easy to check that $\hat{M}_n$ is a diagonally dominant tridiagonal M-matrix and $\hat{E}_n \ge 0$ is a matrix whose elements within the tridiagonal band are zero.

As shown in the table, Bouchon's bounds, although sufficient, greatly underestimate the allowable size of v. This is mainly due to the large size of ω. Indeed, Bouchon's easy-to-apply result gives tighter bounds for small values of ω, i.e., for perturbations near the main diagonal, as is often the case in the context of the analysis of schemes for partial differential equations.

We see from Table 5.1 that our computed bounds are identical to those by Buffoni when an accurate search is performed. Which algorithm is more efficient depends on the specified tolerance during our search procedure. Finally, we note that there is a slight difference between Buffoni's bounds in Table 5.1 and those cited in [6]. This may be due, in part, to differences in the routines used to compute the matrix inverses.

6. Conclusions. In this paper, we have considered perturbations of M-matrices which destroy the M-matrix sign pattern, and have provided bounds on these perturbations so that the perturbed matrix is monotone. Specifically, we considered a general single element perturbation of a tridiagonal M-matrix and have found the maximum allowable perturbation which preserves monotonicity. The bounds obtained are improvements over those found in [16, 15] and those provided by a special case of a general result due to Bouchon [3]. Unlike our previous work, which expressed the sufficient bounds in terms of quantities that characterize the decay of the inverse elements, our new bounds are written explicitly in terms of the elements of the (unperturbed) tridiagonal M-matrix. When simultaneously perturbing more than one element, implicitly defined, necessary, and sufficient conditions are obtained. An algorithm for computing the actual upper bound, suggested by the result, is presented. Numerical experiments were used to demonstrate the efficacy of our bounds, and comparisons are made to existing results in the literature. Finally, the results were applied to an application problem arising from the discretization of an integro-differential equation. Improved sufficient bounds on multiple element perturbations, which preserve monotonicity and are easily computable, in the spirit of Bouchon's work, are currently under investigation.

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